ON THE THEORY OF ABSOLUTELY ADDITIVE DISTRIBUTION FUNCTIONS

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ON THE THEORY OF ABSOLUTELY ADDITIVE DISTRIBUTION FUNCTIONS.

By E. K. HAVILAND.

Introduction. Recently * the author considered the problem of the asymptotic distribution of a movement in an n-space, e. g., the distribution of a complex-valued almost-periodic function ($n = 2$).† The methods used are those of classical probability theory. In particular, it turns out that in the case of the distribution of the Riemann ζ-function ‡ and similar problems in the analytic theory of numbers, the distribution (or repartition) function § is built up, in the sense of classical probability theory, from statistically independent two-dimensional distributions, each of which represents the distribution of an individual term of the almost-periodic development under consideration.

The purpose of the present paper is to develop, without reference to movements, an exact treatment of general multi-dimensional statistical methods which since the time of Laplace have rested upon a purely formal foundation.¶ Although in one dimension the results are already well established,‖ the extension of the theorems in question to more than one dimension is, in general, by no means immediate, as will be explained later. In fact, extensive use will be made of the Radon theory of monotone absolutely additive set functions,** which are considered as defining distributions in n-dimensional space, so that the present paper may be regarded as a contribution to the theory of these functions and certain integrals associated with them. It may be pointed out that it is not, in general, permissible to suppose that

* Cf. E. K. Haviland, *loc. cit.*, I and II. (References appear at the end of this article.) In II, $\lim_{T=\infty} \int_{-T}^{T} \ldots dt$ should obviously be replaced by $\lim_{T=\infty} \int_{-T}^{T} \ldots dt / 2T$.

† The corresponding problem for $n = 1$ has been treated by A. Wintner, *loc. cit.*, I, especially pp. 156-157, and III. The extension to the case $n > 1$ is, however, not immediate.

‡ Cf. A. Wintner, *loc. cit.*, IV, especially pp. 328-329, where references to the literature are given.

§ As defined, e. g., by the author, *loc. cit.*, I, p. 554.

¶ Cf., e. g., J. Hadamard, *op. cit.*, §§ 430-433.

‖ With perhaps the exception of the analogue of Theorem V of the present paper, for which I am not aware of an exact general published proof.

** Cf. J. Radon, *loc. cit.*, I, especially pp. 1295-1342, and *loc. cit.*, II, especially pp. 1092-1094.

Radon distributions may be represented by means of Lebesgue integrals, i. e., that they are absolutely continuous, as, e. g., it is clear that the partial distribution functions mentioned above in connection with the Riemann ζ-function are not absolutely continuous, and in some other applications there occur even "point spectra," where the Radon distribution is not only not absolutely continuous, but not even continuous.

It has previously been shown * that in the case of distribution functions which are zero for all sets lying outside an arbitrarily large but fixed rectangle the Radon momentum problem is determined, in the sense that two distribution functions having the same momenta are identical save at most on their discontinuity lines. Furthermore, if a sequence of distribution functions ϕ_n is such that the momenta approach a limit as n becomes infinite, then $\{\phi_n\}$ converges to a distribution function, provided all ϕ_n are zero for sets lying outside a sufficiently large but fixed rectangle.† In the first part of the present paper, it will be shown that the foregoing restriction on the distribution functions can be removed, and it turns out that the question is, as in the one-dimensional case,‡ rather closely connected with the problem of determinateness in the sense just mentioned. For the determination of the momentum problem, a sufficient and "almost" necessary condition will be obtained.

The second portion of the paper concerns the symbolical product § of two general distribution functions, corresponding to the addition of statistically independent distributions. The existence of this symbolical product is established and it is shown that the Fourier transform of the symbolical product of two distribution functions is the product of the Fourier transforms of the individual functions. In a recent paper,¶ S. Bochner proves by methods differing substantially from those here used that the product of two characteristic functions is again a characteristic function, where by a characteristic function is meant the Fourier transform of a distribution function. He establishes the existence of the symbolical product and proves the relation mentioned above between the symbolical product and Fourier transforms only in the special case that one of the distribution functions is absolutely continuous, so that it can be represented as a Lebesgue integral. As we have pointed out above, such a representation is not always possible, so that the

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* Cf. E. K. Haviland, *loc. cit.*, I, pp. 550-551. Also C. A. Fischer, *loc. cit.*, p. 40 and T. H. Hildebrandt and I. J. Schoenberg, *loc. cit.*, p. 324.

† Cf. E. K. Haviland, *loc. cit.*, I, p. 552. The proof for a sequence of distribution functions is the same as that for a continuum.

‡ Cf. A. Wintner, *loc. cit.*, II; M. Fréchet and J. Shohat, *loc. cit.*

§ Also termed "Faltung" or "convolution."

¶ Cf. S. Bochner, *loc. cit.*

existence of the symbolical product in the general case must be based, as in Theorem IV of the present paper, on a detailed consideration of Radon integrals.

The final portion of this paper treats of the spectra of distribution functions and affords an illustration of some of the important differences between distribution functions in one dimension and those in more than one dimension.

We shall suppose for definiteness that $n = 2$, where n is the number of dimensions of the space in which our set functions are defined, as our discussions admit immediate extension to space of a higher number of dimensions. The case $n = 2$, on the other hand, is for several reasons essentially more complicated than the case $n = 1$. For instance, in the two-dimensional case the discontinuities of the set function do not arise solely from the so-called "point spectrum." Moreover, in one dimension, the set function appears disguised as a point function, so that its true rôle is not as clear as in Radon integrals, at least if the latter are defined over sets more complicated than rectangles.

1. *The momenta of distribution functions.*

Definition 1. The monotone absolutely additive set function * $\phi(E)$ is said to be a distribution function if $0 \leqq \phi(E) \leqq 1$ and

$$\phi(S) = \int \cdots \int_S d_{x_1} \cdots {}_{x_n}\phi(E) = 1,$$

where S denotes the whole n-space.

For theoretical purposes, of course, we are interested in distribution functions primarily insofar as they are monotone absolutely additive set functions of bounded total variation such that any set of them are of uniformly bounded total variation, and our theorems are essentially theorems on functions with these properties.

If, in addition, $\phi(E) = 0$ for all sets E outside a fixed finite rectangle K, we shall say that $\phi(E)$ is damped. In the present paper, we shall not suppose distribution functions damped unless this fact is explicitly stated.

By reasoning precisely similar to that used by Radon in the case of set functions defined for sets lying in a finite rectangle,† we may assign to $\phi(E)$ a point function defined by $F(\xi, \eta) = \phi(R_{\xi\eta})$, where $R_{\xi\eta}$: $(-\infty < x < \xi;$ $-\infty < y < \eta)$. Then it follows,‡ since ϕ is a distribution function, that F,

* For definition, cf. J. Radon, *loc. cit.*, I, p. 1299.

† Cf. J. Radon, *loc. cit.*, I, pp. 1304-1305.

‡ Cf. J. Radon, *loc. cit.*, II, p. 1093.

and hence by definition ϕ, is continuous save at most for points on a finite or denumerable set of lines $x = \xi_i$, $y = \eta_j$ corresponding to the discontinuities of $F(x, +\infty)$, $F(+\infty, y)$ respectively. These lines $x = \xi_i$, $y = \eta_j$ are called singular lines of the distribution function ϕ.

Definition 2. A rectangle is said to be a singular rectangle of ϕ if at least one of its sides is a singular line of ϕ. By a rectangle, singular or non-singular, we shall always mean a rectangle with sides parallel to the coördinate axes.

It is clear that if R is a non-singular rectangle of ϕ and R_1 a second rectangle all of whose sides are sufficiently close to the sides of the former, then $| \phi(R_1) - \phi(R) | < \epsilon$.

Remark. The singular lines or singular rectangles of an arbitrary absolutely additive set function are the singular lines or singular rectangles of the monotone set functions representing its total positive and its total negative variation.*

Definition 3. A sequence of distribution functions $\{\phi_n(E)\}$ is said to converge to a distribution function $\phi(E)$ if $\lim_{n=\infty} \phi_n(R) = \phi(R)$ for all non-singular rectangles R of ϕ.

Definition 4. A sequence of rectangles $\{R_n\}$ is said to be everywhere dense if for every given rectangle $R : (x^1 < x < x^2;\ y^1 < y < y^2)$, not necessarily belonging to the sequence, and for every $\epsilon > 0$ there exists an n such that for the corresponding R_n we have

$$| x^1 - x^1{}_n |, \quad | x^2 - x^2{}_n |, \quad | y^1 - y^1{}_n |, \quad | y^2 - y^2{}_n |$$

all less than ϵ.

If ϕ^{I} and ϕ^{II} are two distribution functions such that $\phi^{\mathrm{I}}(R_n) = \phi^{\mathrm{II}}(R_n)$ on an everywhere dense sequence of rectangles $\{R_n\}$, then $\phi^{\mathrm{I}}(E) = \phi^{\mathrm{II}}(E)$ for every rectangle non-singular with respect to ϕ^{I} and ϕ^{II}. Furthermore, ϕ^{I} and ϕ^{II} have the same singular lines, hence the same singular rectangles.† In what follows, we shall regard two distribution functions as the same and call them essentially identical if their corresponding point functions are identical save at most on their singular lines.

The Fourier transform $\Lambda(s, t; \phi)$ of the distribution function $\phi(E)$ is defined by

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* Cf. E. K. Haviland, *loc. cit.*, I, p. 550.

† The proof is effectively the same as in the damped case. Cf. E. K. Haviland, *loc. cit.*, I, pp. 549-550.

$$(1) \qquad \Lambda(s, t; \phi) = \int\int_S \exp\,[i(sx + ty)]\, d_{xy}\phi(E),$$

where s, t, x, y are all real and S denotes the entire xy-plane.

By the integral $\int\int_S f(x, y)\, d_{xy}\phi(E)$ of a continuous function $f(x, y)$ with respect to the distribution function ϕ is meant the limit as $R \to S$ of $\int\int_R f(x, y)\, d_{xy}\phi(E)$, R being an arbitrarily large rectangle, provided that this limit exists. The integral over R exists by a theorem of Radon * and the limit of this integral as $R \to S$ clearly exists for every bounded function f, inasmuch as ϕ is of finite total variation. In particular, $\Lambda(s, t; \phi)$ exists for all real values of s and t.

Definition 5. The Radon momentum problem associated with a distribution function $\phi(E)$ is said to be determined provided that all momenta

$$\mathrm{M}(p, q; \phi) = \int\int_S x^p y^q d_{xy}\phi(E)$$

exist and are such that any other distribution function ψ for which $\mathrm{M}(p, q; \psi) = \mathrm{M}(p, q; \phi)$; $p, q = 0, 1, 2, \cdots$, must be essentially identical with ϕ.

The author has previously shown that if, as n becomes infinite, $\lim \mathrm{M}(p, q; \phi_n)$ exists for all non-negative integers p, q, and for a sequence of uniformly damped distribution functions $\phi_n(E)$, for which the momentum problem is necessarily determined, then $\lim_{n \to \infty} \phi_n(R) = \phi(R)$, whence $\phi(E)$ is a damped distribution function for which R is a non-singular rectangle. In case the momentum problem for the limiting function ϕ is determined, a situation for which a sufficient and "almost" necessary condition will be given below, the foregoing result may be extended to the case of non-damped distribution functions. The first step in the extension is given by

LEMMA I.† *Let there be given a sequence of distribution functions* $\{\psi_n(E)\}$ *with the following properties:*

(A) *The momenta* $\mathrm{M}(p, q; \psi_n)$ *of all orders* $\mathrm{p} = 0, 1, 2, \cdots$;

* The integral is that defined by J. Radon, *loc. cit.*, I, pp. 1322-1324. This is an extension of the Riemann-Stieltjes integral in one dimension. For the latter, cf. E. W. Hobson, *op. cit.*, pp. 506-516 and H. Lebesgue, *op. cit.*, pp. 252-313.

† These propositions represent the extension to n dimensions, $n > 1$, of results obtained in the one-dimensional case by A. Wintner, *loc. cit.*, II, and by M. Fréchet and J. Shohat, *loc. cit.*

$q = 0, 1, 2, \cdots$ exist for $n = 1, 2, \cdots$ (or at least from a certain rank n on, possibly depending on p, q).

(B) *The quantities $\mathbf{M}(p, q; \psi_n)$ for any fixed p, q lie, when they exist, between two fixed limits independent of n (but possibly dependent on p and q).*

Then there exists a subsequence $\{\phi_i(E) = \psi_{n_i}(E)\}$ such that

(i) *As $i \to \infty$, $\lim \mathbf{M}(p, q; \phi_i)$ exists $(= \mu_{pq}$, say) for all non-negative values of p, q, and*

(ii) *$\{\phi_i(R)\}$ converges to $\phi(R)$, where $\phi(E)$ is a distribution function, on all non-singular rectangles R of ϕ.*

Proof. In virtue of the uniform boundedness of the $\mathbf{M}(p, q; \psi_n)$ for all fixed p and q, from $\{\mathbf{M}(0, 0; \psi_n)\}$ a subsequence $\{\mathbf{M}(0, 0; \psi_{l_j})\}$ having a limit μ_{00} may be selected. From $\{\mathbf{M}(1, 0; \psi_{l_j})\}$ a subsequence $\{\mathbf{M}(1, 0; \psi_{m_j})\}$ converging to a limit μ_{10} may then be selected. Proceeding in this way, we obtain by the diagonal principle of Cantor a sequence $\{\mathbf{M}(p, q; \psi_{l_1}), \mathbf{M}(p, q; \psi_{m_2}), \cdots\}$ converging to a finite limit μ_{pq} for all fixed non-negative values of p, q, where the labels $l_1, m_2, \cdots$ are independent of p, q. The corresponding distribution functions $\gamma_1(E) = \psi_{l_1}(E)$, $\gamma_2(E) = \psi_{m_2}(E), \cdots$ are seen to satisfy (i).

By applying the Compactness Theorem of Radon * to the sequence $\{\gamma_n(E)\}$, we extract from it a subsequence $\{\phi_i(E)\}$ converging to the monotone absolutely additive set function $\phi(E)$ on every non-singular rectangle of the latter.

It remains to be shown that ϕ is actually a distribution function, i. e., that $\phi(S) = 1$. To this end we consider the square $R : (-a < x < a; -a < y < a)$, where $a > 1$. $S - R$ may be divided into two (disconnected) parts $R_1 : (|x| \geqq a; -\infty < y < +\infty)$ and $R_2 : (-a < x < a; |y| \geqq a)$ such that in the former $|x| \geqq a$ while in the latter $|y| \geqq a$. Then by Schwarz' inequality,

$$\left[\int\int_{R_1} x^p y^q d_{xy}\psi_n(E)\right]^2 \leqq \int\int_{R_1} x^{2p} y^{2q} d_{xy}\psi_n(E) \cdot \int\int_{R_1} d_{xy}\psi_n(E)$$
$$\leqq \int\int_{R_1} x^{2p} y^{2q} d_{xy}\psi_n(E) \leqq \mathbf{M}(4p + 2, 2q; \psi_n)/a^{2p+2}$$

and similarly

$$\left[\int\int_{R_2} x^p y^q d_{xy}\psi_n(E)\right]^2 \leqq \mathbf{M}(2p, 4q + 2; \psi_n)/a^{2q+2},$$

* Cf. J. Radon, *loc. cit.*, I, p. 1337. A statement of these theorems more convenient for our purposes is given by the author, *loc. cit.*, I, pp. 551-552. The proof of the Compactness Theorem is there given for a finite rectangle but may be extended to the whole plane by the diagonal process of Cantor.

these appraisals being obviously analogous to a device of Tschebycheff.* If now we choose as our $\{\psi_n(E)\}$ the sequence $\{\phi_i(E)\}$ converging to $\phi(E)$, we may infer for i sufficiently large that

$$0 \leqq \mathrm{M}(4p+2, 2q; \phi_i) < \mu_{4p+2,2q} + 1,$$

and

$$0 \leqq \mathrm{M}(2p, 4q+2; \phi_i) < \mu_{2p,4q+2} + 1.$$

If, in particular, we set $p = q = 0$, we obtain

$$\Big[\int\int_{R_1} d_{xy}\phi_i(E)\Big]^2 < (\mu_{20}+1)/a^2 \text{ and } \Big[\int\int_{R_2} d_{xy}\phi_i(E)\Big]^2 < (\mu_{02}+1)/a^2.$$

But μ_{20} and μ_{02} are from their definitions non-negative, so

$$0 \leqq \int\int_{R_1} d_{xy}\phi_i(E) < (\mu_{20}+1)^{1/2}/a \text{ and } 0 \leqq \int\int_{R_2} d_{xy}\phi_i(E) < (\mu_{02}+1)^{1/2}/a.$$

Hence

$$0 \leqq \int\int_{S-R} d_{xy}\phi_i(E) < [(\mu_{20}+1)^{1/2} + (\mu_{02}+1)^{1/2}]/a.$$

As $\int\int_{S-R} = \int\int_S - \int\int_R$ and $\phi_i(E)$ is a distribution function, the preceding inequality is equivalent to

$$0 \leqq 1 - \phi_i(R) < [(\mu_{20}+1)^{1/2} + (\mu_{02}+1)^{1/2}]/a.$$

Letting i become infinite, we see that

$$0 \leqq 1 - \phi(R) \leqq [(\mu_{20}+1)^{1/2} + (\mu_{02}+1)^{1/2}]/a,$$

so that

$$\lim_{a=\infty} \phi(R) = \phi(S) = 1, \qquad \text{q. e. d.}$$

Remark. We shall later have occasion to make use of the fact that for any given non-negative values of p and q, the convergence of

$$\int\int_R x^p y^q d_{xy}\psi_n(E) \text{ to } \int\int_S x^p y^q d_{xy}\psi_n(E)$$

is uniform with respect to n, at least from a certain rank n on.

Proof. As in the proof of the preceding lemma,

$$\Big|\int\int_{S-R} x^p y^q d_{xy}\psi_n(E)\Big| \leqq \Big|\int\int_{R_1} x^p y^q d_{xy}\psi_n(E)\Big| + \Big|\int\int_{R_2} x^p y^q d_{xy}\psi_n(E)\Big|$$
$$\leqq [\mathrm{M}(4p+2, 2q; \psi_n)]^{1/2}/a^{p+1} + [\mathrm{M}(2p, 4q+2; \psi_n)]^{1/2}/a^{q+1}$$
$$\leqq [\mathrm{M}_{4p+2,2q}]^{1/2}/a^{p+1} + [\mathrm{M}_{2p,4q+2}]^{1/2}/a^{q+1},$$

* Cf. P. L. Tschebycheff, *loc. cit.*, p. 691 and A. Kolmogoroff, *op. cit.*, p. 37.

since by condition (B) of the lemma

$$0 \leqq \mathrm{M}(4p+2, 2q; \psi_n) \leqq \mathsf{M}_{4p+2,2q} \text{ and } 0 \leqq \mathrm{M}(2p, 4q+2; \psi_n) \leqq \mathsf{M}_{2p,4q+2},$$

q. e. d.

We are now in a position to prove

THEOREM I (MOMENTUM THEOREM). *Let there be given a sequence of distribution functions* $\{\phi_n(E)\}$ *satisfying conditions* (A) *and* (B) *of Lemma I. Then if* $\lim_{n=\infty} \mathrm{M}(p, q; \phi_n)$ *exists,* $= \mu_{pq}$ *say, for all non-negative values of* p *and* q, *there exists at least one distribution function, say* $\phi(E)$, *such that* $\mu_{pq} = \mathrm{M}(p, q; \phi)$, *and a subsequence* $\{\phi_{n_i}(E)\}$ *can be extracted from the given sequence of distribution functions so that* $\lim_{i=\infty} \phi_{n_i}(R) = \phi(R)$ *for all non-singular rectangles* R *of* $\phi(E)$.

If, in addition, the sequence $\{\mu_{pq}\}$ *is such that* ϕ *is uniquely determined by it, then the sequence* $\{\phi_n(E)\}$ *itself converges as* n *becomes infinite to* $\phi(R)$ *on any non-singular rectangle* R *of* ϕ.

Proof. By Lemma I, the existence of the subsequence $\{\phi_{n_i}\}$ converging to a ϕ is assured.

Now if a sequence of distribution functions $\{\phi_n(E)\}$ is such that

(a) $\lim_{n=\infty} \phi_n(Q) = \phi(Q)$ for every non-singular rectangle Q of the distribution function ϕ, and

(b) $\iint_R f(x, y)\, d_{xy}\phi_n(E) \to \iint_S f(x, y)\, d_{xy}\phi_n(E)$ uniformly with respect to n as $R \to S$, $f(x, y)$ being continuous in every finite rectangle R, then the Radon Term by Term Integration Theorem holds for the infinite domain S, so that

$$\lim_{n=\infty} \iint_S f(x, y)\, d_{xy}\phi_n(E) = \iint_S f(x, y)\, d_{xy}\phi(E). \tag{2}$$

For if R'' be a closed rectangle and R' an open rectangle lying in the interior of R'', it follows from the Term by Term Integration Theorem for a finite domain * that

$$\left| \iint_{R''-R'} f(x, y)\, d_{xy}\phi(E) \right| = \left| \lim_{n=\infty} \iint_{R''-R'} f(x, y)\, d_{xy}\phi_n(E) \right|$$

* Cf. J. Radon, *loc. cit.*, I, p. 1337. A statement of these theorems more convenient for our purposes is given by the author, *loc. cit.*, I, pp. 551-552. The proof of the Compactness Theorem is there given for a finite rectangle but may be extended to the whole plane by the diagonal process of Cantor.

and the right-hand side of this equation is $< \epsilon$ for R' sufficiently large by (b). Hence the integral on the right of (2) exists. Furthermore,

$$\left| \int\int_S f(x,y)\, d_{xy}\phi(E) - \int\int_S f(x,y)\, d_{xy}\phi_n(E) \right| \leqq \left| \int\int_{S-R} f(x,y)\, d_{xy}\phi(E) \right|$$
$$+ \left| \int\int_{S-R} f(x,y)\, d_{xy}\phi_n(E) \right| + \left| \int\int_R f(x,y)\, d_{xy}\phi(E) - \int\int_R f(x,y)\, d_{xy}\phi_n(E) \right| < \epsilon$$

if R be taken sufficiently large and the theorem then applied for a finite domain. Moreover, by the Remark following Lemma I, the foregoing conditions (a) and (b) are satisfied if we replace $\{\phi_n(E)\}$ by $\{\phi_{n_i}(E)\}$ and $f(x,y)$ by $x^p y^q$. Hence

$$\mu_{rs} = \lim_{n=\infty} \mathbf{M}(p,q;\phi_n) = \lim_{i=\infty} \mathbf{M}(p,q;\phi_{n_i}) = \lim_{i=\infty} \int\int_S x^p y^q d_{xy}\phi_{n_i}(E)$$
$$= \int\int_S x^p y^q d_{xy}\phi(E).$$

This proves the first paragraph of the present theorem. The last paragraph is proved as follows:

Suppose there exists a non-singular rectangle R_0 of ϕ such that $\{\phi_n(R_0)\}$ does not converge to $\phi(R_0)$. Then a subsequence $\{\gamma_k(R_0) = \phi_{n_k}(R_0)\}$ can be extracted such that as k becomes infinite $\{\gamma_k(R_0)\}$ converges to a certain number $A \neq \phi(R_0)$. On the other hand, we have seen that the sequence $\{\gamma_k(E)\}$ gives rise to a subsequence $\{\delta_j(E)\}$ which converges as $j \to \infty$ to a distribution function $\delta(E)$ on all non-singular rectangles of the latter. Now $\delta(E)$ has the same momenta μ_{pq} as has $\phi(E)$, and therefore, since by hypothesis the momentum problem corresponding to $\{\mu_{pq}\}$ is determined,* $\lim \delta_j(R_0) = \delta(R_0) = \phi(R_0)$ when $j \to \infty$, since R_0 is a non-singular rectangle of ϕ. This contradicts the fact that $\{\delta_j(R_0)\}$ is a subsequence of $\{\gamma_k(R_0)\}$, which converges, but not to $\delta(R_0)$ and hence completes the proof of the theorem.

The condition that the momentum problem corresponding to $\{\mu_{pq}\}$ be determined is not only sufficient for the limiting relation $\lim \phi_n(E) = \phi(E)$ as $n \to \infty$ on any non-singular rectangle of the latter, but it is necessary as well. For if $\phi(E)$ and $\psi(E)$ be two distinct solutions of the momentum problem in question, then $\{\phi_n(E)\}$ should converge simultaneously to $\phi(E)$ and $\psi(E)$ at all non-singular rectangles of ϕ and ψ. But this is impossible,

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* The momentum problem associated with $\{\mu_{pq}\}$ is said to be determined if

$$\mu_{pq} = \int\int_S x^p y^q d_{xy}\phi^{\mathrm{I}}(E) = \int\int_S x^p y^q d_{xy}\phi^{\mathrm{II}}(E)\,;\ p, q, = 0, 1, 2, \cdots,$$

implies ϕ^{I} and ϕ^{II} are essentially identical.

as there is at least one non-singular rectangle R_0 common to ϕ and ψ such that $\phi(R_0) \neq \psi(R_0)$.

In view of the rôle thus played by the determinateness of the momentum problem, we prove

THEOREM II.* *A sufficient condition that a distribution function be determined by its momenta $G_{pq} = \mathbf{M}(p, q; \phi)$ is that*

$$\left(\sum_{\nu=0}^{2m} \binom{2m}{\nu} \left| G_{2m-\nu,\nu} \right|\right)^{1/(2m)} = o(m). \tag{3}$$

This sufficient condition is, furthermore, almost necessary in that ϕ_1 need not equal ϕ_2 if $o(m)$ is replaced by $o(m^{1+\epsilon})$.

Proof. Let $\phi_1(E)$, $\phi_2(E)$ be distribution functions. Then as these functions are uniquely determined by their Fourier transforms,† the momentum problem is determined under conditions such that $\Lambda(s, t; \phi_1) = \Lambda(s, t; \phi_2)$, where s, t, x, y are all real.

Inasmuch as the formula

$$f(x+h) = f(x) + hf'(x) + \cdots + h^{n-1}(n-1)!^{-1}f^{(n-1)}(x) + (n-1)!^{-1}\int_0^h f^{(n)}(x+t)(h-t)^{n-1}dt$$

is obtained merely by successive partial integrations of the integral occurring in the remainder term, the formula is valid for complex functions of a real variable also, so that we obtain in the present case

$$e^{i\xi} = 1 + i\xi + \cdots + (i\xi)^{n-1}(n-1)!^{-1} + f_n(\xi) \tag{4}$$

where

$$|f_n(\xi)| = \left| i^n(n-1)!^{-1}\int_0^\xi e^{it}(\xi-t)^{n-1}dt \right| \leqq (n-1)!^{-1}\int_0^\xi |(\xi-t)|^{n-1}dt = |\xi|^n/n!$$

Setting $n = 2m$ and $\xi = sx + ty$ in (4), integrating first with respect to ϕ_1, then with respect to ϕ_2 and subtracting, we obtain, since all momenta are equal by hypothesis,

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* The present proof of the sufficient condition is based on a treatment by F. P. Cantelli of the one-dimensional case under the assumption of absolutely continuous distribution functions. Cf. F. P. Cantelli, *loc. cit.*, pp. 155-157. In the one-dimensional case it is known that the theorem holds even if $o(m)$ and $o(m^{1+\epsilon})$ are replaced by $O(m)$. Cf. T. Carleman, *op. cit.*, p. 81.

† Cf. S. Bochner, *loc. cit.*, p. 402, Theorem 14. Another proof of this theorem will be given below.

$$
\begin{aligned}
&| \Lambda(s,t;\phi_1) - \Lambda(s,t;\phi_2) | \\
&= \Big| \int\int_S \exp[i(sx+ty)]\, d_{xy}\phi_1(E) - \int\int_S \exp[i(sx+ty)]\, d_{xy}\phi_2(E) \Big| \\
&\leqq \int\int_S | f_{2m}(sx+ty) |\, d_{xy}\phi_1(E) + \int\int_S | f_{2m}(sx+ty) |\, d_{xy}\phi_2(E) \\
&\leqq (2m)!^{-1} \int\int_S (sx+ty)^{2m} d_{xy}\phi_1(E) + (2m)!^{-1} \int\int_S (sx+ty)^{2m} d_{xy}\phi_2(E) \\
&= 2[(2m)!]^{-1} \sum_{\nu=0}^{2m} \binom{2m}{\nu} s^{2m-\nu} t^{\nu} G_{2m-\nu,\nu},
\end{aligned}
$$

so that on placing $b = \max(|s|, |t|)$, where s and t are fixed,

$$\Lambda(s,t;\phi_1) = \Lambda(s,t;\phi_2) \text{ if, for a fixed value of } b,$$

$$b^{2m}(2m)!^{-1} \sum_{\nu=0}^{2m} \binom{2m}{\nu} | G_{2m-\nu,\nu} | = o(1),$$

hence, a fortiori, if

$$(2m)!^{-1/(2m)} \left[\sum_{\nu=0}^{2m} \binom{2m}{\nu} \Big| G_{2m-\nu,\nu} \Big| \right]^{1/(2m)} = o(1),$$

a condition implied by (3), inasmuch as by the Stirling Formula

$$(2m)!^{1/(2m)} = O(m).$$

As applications of the foregoing determinateness condition, we consider

1. The case in which the distribution function is damped.* In this case,

$$| G_{2m-\nu,\nu} | = \Big| \int\int_S x^{2m-\nu} y^{\nu} d_{xy}\phi(E) \Big| \leqq R^{2m},$$

provided $\phi(E) = 0$ for all sets E outside $(-R < x < R; -R < y < R)$. Hence

$$\left[\sum_{\nu=0}^{2m} \binom{2m}{\nu} \Big| G_{2m-\nu,\nu} \Big| \right]^{1/(2m)} \leqq 2R = o(m),$$

so that the condition (3) is satisfied.

2. The case in which

$$\phi(E) = \int\int_E f(x,y)\, dxdy, \text{ where } 0 \leqq f(x,y) = O(\exp[-(x^2+y^2)^{\alpha}]),\ 0 < \alpha,\ (x^2+y^2 \to +\infty).$$

It is easy to see that (3) is satisfied by the momenta G_{pq} of such a ϕ if

$$\left[\sum_{\nu=0}^{2m} \binom{2m}{\nu} \Big| H_{2m-\nu,\nu} \Big| \right]^{1/(2m)} = o(m) \tag{5}$$

* For a direct treatment of this case, cf. E. K. Haviland, *loc. cit*, I, pp. 552-553.

where

$$H_{pq} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^p y^q \exp\left[-(x^2+y^2)^\alpha\right] dx\,dy, \qquad (0<\alpha).$$

Now

$$\sum_{\nu=0}^{2m}\binom{2m}{\nu}\left|H_{2m-\nu,\nu}\right| \leqq \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(|x|+|y|)^{2m}\exp\left[-(x^2+y^2)^\alpha\right]dx\,dy$$

$$= 4\int_0^{+\infty}\int_0^{+\infty}\left[(x+y)^2\right]^m \exp\left[-(x^2+y^2)^\alpha\right]dx\,dy$$

$$\leqq 2^{m+2}\int_0^{+\infty}\int_0^{+\infty}(x^2+y^2)^m\exp\left[-(x^2+y^2)^\alpha\right]dx\,dy$$

$$= 2^{m+2}\int_0^{\pi/2}d\vartheta\int_0^{+\infty} r^{2m+1}\exp\left[-r^{2\alpha}\right]dr$$

$$= 2^m\pi\Gamma([m+1]/\alpha)/\alpha.$$

The $2m$-th root of this expression is * $= O(m^{1/(2\alpha)})$, so that (5), hence (3), is satisfied if $\alpha > \frac{1}{2}$ but not if $\alpha \leqq \frac{1}{2}$. Consequently, for $\alpha > \frac{1}{2}$, we have determinateness. It is to be noticed that this includes the classical case $\alpha = 1$.

Furthermore, the sufficient condition (3) is almost necessary. To show this, we first observe that the one-dimensional distribution function may be regarded as a special case of the two-dimensional. For suppose $\sigma(x)$ a one-dimensional distribution function along X : $(-\infty < x < +\infty;\ y = 0)$, i. e., $\sigma(x)$ is a monotone non-decreasing function of x there and $\sigma(-\infty) = 0$, $\sigma(+\infty) = 1$. Then we can define a set function $\phi_1(R)$ for all (half-open) rectangles R of the xy-plane by setting $\phi_1(R) = \sigma(X\cdot R) = \sigma(x_2) - \sigma(x_1)$, x_1 and x_2 $(x_2 \geqq x_1)$ being the end points of the segment $X\cdot R$, if R contains a portion of the x-axis; $\phi_1(R) = 0$ otherwise. It is known † that ϕ_1 determines a monotone absolutely additive set function ϕ uniquely up to the singular lines of the latter. Moreover, $\phi_1(R) = \phi(R)$ on every non-singular rectangle R of ϕ, from which fact it follows, in particular, that $\phi(S) = 1$, so that ϕ is a distribution function.

Let M : $(-K \leqq x \leqq K;\ -K \leqq y \leqq K)$ be an arbitrarily large square in the plane and let U : $(-\infty < x < +\infty;\ -\epsilon \leqq y \leqq \epsilon)$, $0 < \epsilon < 1$,

* Cf. E. Artin, *op. cit.*, p. 23, equation (26).

† This follows from the reasoning of J. Radon, *loc. cit.*, I, pp. 1305-1313 and pp. 1338-1339 if it be observed that to a monotone additive interval function there may be made to correspond a monotone point function whose continuity points are everywhere dense. Cf. also S. Bochner, *loc. cit.*, pp. 387-388; E. K. Haviland, *loc. cit.*, III, Note 3.

be a strip enclosing the x-axis. Furthermore, we denote $M \cdot U$ by V. Then by the inequality of Schwarz,

$$\left[\int\int_M x^p y^q d_{xy}\phi(E)\right]^2 \leqq \int\int_M x^{2p} y^{2q} d_{xy}\phi(E) \cdot \int\int_M d_{xy}\phi(E)$$
$$\leqq \int\int_{M-V} x^{2p} y^{2q} d_{xy}\phi(E) + \int\int_V x^{2p} y^{2q} d_{xy}\phi(E).$$

But the integral over $M - V$ is zero by the definition of ϕ, and

$$\int\int_V x^{2p} y^{2q} d_{xy}\phi(E) \leqq \epsilon^{2q} \int\int_S x^{2p} d_{xy}\phi(E) = \epsilon^{2q} \mathrm{M}(2p, 0; \phi).$$

It follows that $\int\int_M x^p y^q d_{xy}\phi(E) < \eta$ if $q > 0$, or as M is arbitrarily large and η arbitrarily small,

$$\text{(6a)} \qquad \mathrm{M}(p, q; \phi) = 0, \qquad p = 0, 1, 2, \cdots; \; q = 1, 2, \cdots$$

If $q = 0$, the definition of ϕ shows that

$$\int\int_M x^p d_{xy}\phi(E) = \int_{-K}^{K} x^p d\sigma(x).$$

Hence, as the right-hand side approaches a limit as $M \to S$,

$$\text{(6b)} \qquad \mathrm{M}(p, 0; \phi) = \int\int_S x^p d_{xy}\phi(E) = \int_{-\infty}^{+\infty} x^p d\sigma(x), \; p = 0, 1, 2, \cdots$$

In consequence, the one-dimensional distribution functions represented by

$$\sigma(x) = D_h \int_{-\infty}^{x} \exp\left[-x^\nu\right]\left[1 + h\cos(x^\nu \tan \tfrac{1}{2}\nu\pi)\right] dx;$$

where $$-1 \leqq h \leqq 1, \qquad \nu = 2\beta/(2\beta + 1),$$

β being a positive integer, and

$$D_h^{-1} = \int_{-\infty}^{+\infty} \exp\left[-x^\nu\right]\left[1 + h\cos(x^\nu \tan \tfrac{1}{2}\nu\pi)\right] dx,$$

may be regarded as two-dimensional distribution functions $\phi(E)$ whose momenta are given by

$$\mathrm{M}(p, q; \phi) = 0, \quad q > 0;$$
$$\mathrm{M}(p, 0; \phi) = \int_{-\infty}^{+\infty} x^p \exp\left[-x^\nu\right]\left[1 + h\cos(x^\nu \tan \tfrac{1}{2}\nu\pi)\right] dx.$$

But it is known * that

$$\int_{-\infty}^{+\infty} x^p \exp[-x^\nu] \cos(x^\nu \tan \tfrac{1}{2}\nu\pi)\, dx = 0$$

for every p, so that the momentum problem for such a function $\phi(E)$ is not determined, inasmuch as ϕ depends on h while the momenta do not. In this case, the left-hand side of (3) becomes

$$\begin{aligned}[\,|\,G_{2m0}\,|\,]^{1/(2m)}, \quad \text{where} \quad |\,G_{2m0}\,| &= G_{2m0} = \mathbf{M}(2m, 0\,;\phi) \\ &= \int_{-\infty}^{+\infty} x^{2m} \exp[-x^\nu]dx = 2\int_0^{+\infty} x^{2m}\exp[-x^\nu]dx = 2\Gamma([2m+1]/\nu)/\nu \\ &= o(m^{1+\epsilon}),\end{aligned}$$

where $\epsilon > 0$ may be made arbitrarily small by taking β sufficiently large. As we do not obtain determinateness in this case, it follows that (3) is almost necessary.

In obtaining the above sufficient condition for determinateness, we have made use of a theorem of Bochner † which states that if $\phi_1(E)$ and $\phi_2(E)$ be two distribution functions such that $\Lambda(s, t\,;\phi_1) = \Lambda(s, t\,;\phi_2)$ for all points (s, t) (which is the case if they are equal for a set of points everywhere dense), then $\phi_1(E) = \phi_2(E)$ up to their singular lines. The following proof ‡ of this proposition seems to be more direct than the original and is more closely related to the methods, viz., those of Radon integrals, of the present paper.

By use of the one-dimensional Fourier cosine inversion formula, it may be shown that

$$\begin{aligned}\int_{-\infty}^{+\infty} (\pi\eta s^2)^{-1} \sin\eta s \sin(h+\eta)s \exp[isx]ds &= E_\eta(x, h) = E_\eta(-x, h) \\ &= \begin{cases} 1, \text{ if } 0 \leqq x \leqq h \\ 1 - (x-h)/2\eta, \text{ if } h \leqq x \leqq h + 2\eta \\ 0, \text{ if } h + 2\eta \leqq x \end{cases}\end{aligned}$$

where $h > 0$, $\eta > 0$. Then setting

$$D(s, t\,; x, y) = (\pi^2\eta^2 s^2 t^2)^{-1} \sin\eta s \sin\eta t \sin(h+\eta)s \sin(k+\eta)t \exp[i(sx+ty)],$$

we obtain

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* Cf. W. Stekloff, *loc. cit.*, pp. 50-51.

† Cf. S. Bochner, *loc. cit.*, p. 402, Theorem 14.

‡ The method of this proof has been suggested by the treatment of a similar problem in one dimension by G. Pólya, *loc. cit.*, pp. 107-108.

$$(7)\qquad E_\eta(x,h)E_\eta(y,k)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}D(s,t;x,y)\,ds\,dt=G_\eta(x,y;h,k)$$
$$=\begin{cases}1, & \text{if } (x,y) \text{ is in } I_1,\\ 0, & \text{if } (x,y) \text{ is outside } I_2,\end{cases}$$

where I_1 is the rectangle $(-h\leqq x\leqq h;\ -k\leqq y\leqq k)$ and I_2 is the rectangle $(-(h+2\eta)\leqq x\leqq h+2\eta;\ -(k+2\eta)\leqq y\leqq k+2\eta)$.

Furthermore, $0\leqq G_\eta(x,y;h,k)\leqq 1$ if (x,y) is in I_2-I_1. As $G_\eta(x,y;h,k)$ is the product of two continuous functions of x and y, it is a continuous function of those variables. Finally, we note that although G_η was defined as an iterated integral, it may be regarded as a double integral since the latter exists when taken over any finite rectangle R and is absolutely convergent as $R\to T$, the st-plane.

As $\Lambda(s,t;\phi_i)$, $(i=1,2)$, are continuous * functions of s and t together throughout the st-plane and $|\Lambda(s,t;\phi_i)|\leqq 1$ for all (s,t),

$$\int\int_R D(s,t;-u,-v)\Lambda(s,t;\phi_1)\,dsdt$$
$$=\int\int_R D(s,t;-u,-v)\Lambda(s,t;\phi_2)\,dsdt$$

and hence, as both integrals converge absolutely,

$$(8)\qquad \int\int_T D(s,t;-u-v)\Lambda(s,t;\phi_1)\,dsdt$$
$$=\int\int_T D(s,t;-u-v)\Lambda(s,t;\phi_2)\,dsdt.$$

If we set

$$(9)\qquad H(s,t;x,y)=D(s,t;x-u,\ y-v),$$

these integrals may be separated with the use of two arbitrary non-singular rectangles J, K in the xy-plane and the st-plane respectively into

$$\int\int_K\Big[\int\int_J H(s,t;x,y)\,d_{xy}\phi_i(E)\Big]\,dsdt$$
$$+\int\int_{T-K}\Big[\int\int_J H(s,t;x,y)\,d_{xy}\phi_i(E)\Big]\,dsdt$$
$$+\int\int_T\Big[\int\int_{S-J} H(s,t;x,y)\,d_{xy}\phi_i(E)\Big]\,dsdt,\qquad (i=1,2).$$

* For

$$\int\int_R \exp\,[i(sx+ty)]\,d_{xy}\phi_i(E)$$

is a continuous function of s and t together, R being any finite rectangle, and ϕ_i is a distribution function.

As the last two integrals may be made arbitrarily small in absolute value by taking J and K sufficiently large,

$$(10)\quad \left| \int\int_T \left[\int\int_S H(s,t;x,y)\, d_{xy}\phi_i(E) \right] dsdt - \int\int_K \left[\int\int_J H(s,t;x,y)\, d_{xy}\phi_i(E) \right] dsdt \right| < \tfrac{1}{2}\epsilon$$

for suitable J, K. Similarly, if the non-singular rectangles U and V are taken sufficiently large,

$$(11)\quad \left| \int\int_S \left[\int\int_T H(s,t;x,y)\, dsdt \right] d_{xy}\phi_i(E) - \int\int_U \left[\int\int_V H(s,t;x,y)\, dsdt \right] d_{xy}\phi_i(E) \right| < \tfrac{1}{2}\epsilon \qquad (i=1,2).$$

Since (10) and (11) clearly hold if all four rectangles are sufficiently large, we may choose

$$(12)\qquad K = V \text{ and } J = U.$$

Again, if $f(s,t;x,y)$ be a continuous function of $s,t;x,y$ together in

Σ: $(-M_1 \leqq s \leqq M_1;\ -M_1 \leqq t \leqq M_1;\ -M_2 \leqq x \leqq M_2;\ -M_2 \leqq y \leqq M_2)$,

if Σ_1 and Σ_2 denote the rectangles

$(-M_1 \leqq s \leqq M_1;\ -M_1 \leqq t \leqq M_1)$ and $(-M_2 \leqq x \leqq M_2;\ -M_2 \leqq y \leqq M_2)$,

respectively and if $\phi(E)$ is a monotone absolutely additive set function defined in Σ_2, then

$$(13)\quad \int\int_{\Sigma_1} \left[\int\int_{\Sigma_2} f(s,t;x,y)\, d_{xy}\phi(E) \right] dsdt = \int\int_{\Sigma_2} \left[\int\int_{\Sigma_1} f(s,t;x,y)\, dsdt \right] d_{xy}\phi(E)$$

since both differ arbitrarily little from

$$\sum_{j=1}^{m} \sum_{k=1}^{n} f(s_j, t_j; x_k, y_k)\, \phi(E_k)\, \Delta_j,$$

provided the subdivisions E_k, Δ_j are taken sufficiently fine.

Since the preceding conditions are satisfied by taking $\phi = \phi_i(E)$, $(i=1,2)$, and $f(s,t;x,y) = H(s,t;x,y)$, we obtain, on combining (10), (11) and (13) and taking account of (12):

$$\int\int_T \Big[\int\int_S H(s,t;x,y)\,d_{xy}\phi_i(E)\Big]\,dsdt$$
$$= \int\int_S \Big[\int\int_T H(s,t;x,y)\,dsdt\Big]\,d_{xy}\phi_i(E), \qquad (i=1,2).$$

On combining this result with (9), (8) and (7) we see that

$$\int\int_S G_\eta(x-u, y-v; h, k)\,d_{xy}\phi_1(E) = \int\int_S G_\eta(x-u, y-v; h, k)\,d_{xy}\phi_2(E)$$

or

$$\int\int_S G_\eta(x-u, y-v; h, k)\,d_{xy}\psi(E) = 0, \qquad (14)$$
where $\psi(E) = \phi_1(E) - \phi_2(E)$.

As $\psi(E)$ is an absolutely additive set function of bounded total variation, its non-singular rectangles are everywhere dense.* By a suitable choice of h, k and (u, v) in (7) and (9), the rectangle I_1 may be made to coincide with any non-singular rectangle R of ψ, so that from (14)

$$\int\int_{I_2-I_1} G_\eta(x-u, y-v; h, k)\,d_{xy}\psi(E) + \int\int_{I_1} d_{xy}\psi(E) = 0.$$

As the first integral may be made arbitrarily small by taking η in (7) sufficiently small,

$$\int\int_R d_{xy}\psi(E) = \psi(R) = 0$$

for every non-singular rectangle of $\psi = \phi_1 - \phi_2$.

It follows that $\phi_1(R) = \phi_2(R)$ on all rectangles R non-singular with respect to ψ, ϕ_1, and ϕ_2 and hence (since the singular lines are at most denumerable) on an everywhere dense set of rectangles in the plane. Now it is known that if ϕ_1 and ϕ_2 be two monotone absolutely additive set functions of bounded total variation in the entire plane and such that $\phi_1(R_n) = \phi_2(R_n)$ on the everywhere dense sequence $\{R_n\}$, then $\phi_1(R) = \phi_2(R)$ for every rectangle non-singular with respect to ϕ_1 and ϕ_2. Furthermore, ϕ_1 and ϕ_2 have the same singular lines and the same singular rectangles,† q. e. d.

2. *The symbolical product of distribution functions.* Although the Radon integral as defined ‡ for a continuous integrand has been sufficient for

* The proof is essentially the same as that given by J. Radon, *loc. cit.*, II, p. 1093.

† The proof of this statement is essentially that given by the author, *loc. cit.*, I, p. 550, Theorem II.

‡ The integral is that defined by J. Radon, *loc. cit.*, I, pp. 1322-1324.

our previous purposes, it will not suffice for the considerations to follow. At the same time, it is neither convenient nor necessary to resort to the very general definition (corresponding to Lebesgue integrals) given by Radon * of the integral of functions measurable with respect to an absolutely additive set function. Accordingly, we shall extend the conception of the ordinary Radon integral to the case where the integrand, though bounded, is no longer continuous.

Let ϕ be an absolutely additive set function defined for a set † T of sets in the xy-plane. Let $f(x, y)$ be a bounded single-valued point function defined on the bounded set E_0 belonging to T. Let Π denote a division of E_0 into subsets $E_1, \cdots, E_n$, all belonging to T, and $D(\Pi)$ the greatest of the diameters of the sets $E_1, \cdots, E_n$. Finally, let U_k denote the least upper bound and L_k the greatest lower bound of $f(x, y)$ in E_k. Then if, as $D(\Pi) \to 0$,

$$\lim \sum_{k=1}^{n} U_k \phi(E_k) \quad \text{and} \quad \lim \sum_{k=1}^{n} L_k \phi(E_k)$$

exist and are equal, the Radon integral of f over E_0 with respect to ϕ is said to exist.

Definition 6. By the two-dimensional saltus $\omega(\xi, \eta)$ of $f(x, y)$ at a point (ξ, η) is meant

$$\lim_{\delta=0} [U_\delta(\xi, \eta) - L_\delta(\xi, \eta)],$$

where $U_\delta(\xi, \eta)$, $L_\delta(\xi, \eta)$ are respectively the least upper bound and the greatest lower bound of $f(x, y)$ in $R_\delta(\xi, \eta) : |\xi - x| < \delta, |\eta - y| < \delta$.

If a function $f(x, y)$ is defined in a closed rectangle J, we shall denote by G_ϵ the closed set of points in J at which $\omega(\xi, \eta) \geqq \epsilon$. If (ξ, η) is a boundary point of J, we agree to confine the points (x, y) of Definition 6 to points of J. Using this notation we shall prove

THEOREM III. *Let $f(x, y)$ be bounded and single-valued in*

$$J : (-M \leqq x \leqq M; -M \leqq y \leqq M)$$

and $\phi(E)$ an absolutely additive set function defined in the same region. Then a necessary and sufficient condition that the Radon integral

$$\int\int_{E_0} f(x, y)\, d_{xy}\phi(E)$$

* Cf. J. Radon, *loc. cit.*, I, p. 1329.

† For a definition of this set, cf. J. Radon, *loc. cit.*, I, p. 1299.

exist, where E_0 is any point set in J belonging to the domain of definition of ϕ, is that the total variation of $\phi(E)$ over the open ρ-neighborhood of G_ϵ should for every ϵ approach zero as $\rho \to 0$.*

Proof. We first prove the sufficiency of the condition. For this, one need consider only the case where $\phi(E)$ is monotone, inasmuch as any absolutely additive set function $\phi(E)$ may be expressed as the difference of two monotone absolutely additive set functions † $\psi_1(E)$ and $\psi_2(E)$ and

$$\int\int_{E_0} f(x,y)\, d_{xy}\phi(E) = \int\int_{E_0} f(x,y)\, d_{xy}\psi_1(E) - \int\int_{E_0} f(x,y)\, d_{xy}\psi_2(E).$$

If $u = u(\Pi) = \sum_{k=1}^{m} U_k \phi(E_k)$ and $l = l(\Pi) = \sum_{k=1}^{m} L_k \phi(E_k)$ and if Π, Π' are two divisions of E_0 and $\Pi\Pi'$ the division consecutive to them, then $l(\Pi) \leqq l(\Pi\Pi') \leqq u(\Pi\Pi') \leqq u(\Pi')$, so that the least upper bound of the $l(\Pi)$ is not greater that the greatest lower bound of the $u(\Pi)$. Therefore the integral exists if $|\, u(\Pi) - l(\Pi)\,| < \epsilon$, provided $D(\Pi) < \delta$, where $\epsilon > 0$ is arbitrarily small and $\delta = \delta(\epsilon)$.

Let G_ϵ be as defined above. With centre (x, y), any point of G_ϵ, let a circle of radius 2ρ be taken, and let the open non-overlapping finite set formed by the coalescence of these circular regions be denoted by $N_{\epsilon,2\rho}$. It is the 2ρ-neighborhood of G_ϵ. Under the hypothesis regarding the total variation of $\phi(E)$, ρ can be so chosen that $0 \leqq \phi(\Delta_{\epsilon,2\rho}) < \zeta'$, where $\Delta_{\epsilon,2\rho} = J \cdot N_{\epsilon,2\rho}$. Then by taking the subdivision Π so fine that $D(\Pi) < \rho$, E_0 may (for this subdivision) be divided into two parts, E_0' and E_0'', where E_0' includes all points of E_0 in $N_{\epsilon,\rho}$, the ρ-neighborhood of G_ϵ, but is included in $N_{\epsilon,2\rho}$; and $E_0'' = E_0 - E_0'$. By E_k', E_k'' we denote those E_k composing E_0', E_0'' respectively. Accordingly,

$$u(\Pi) = u'(\Pi) + u''(\Pi), \qquad l(\Pi) = l'(\Pi) + l''(\Pi),$$

where we separate the contributions to the sums made by E_0' and by E_0''. Then

$$u(\Pi) - l(\Pi) = u'(\Pi) - l'(\Pi) + u''(\Pi) - l''(\Pi).$$

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* For the definition of total variation, cf. J. Radon, *loc. cit.*, I, p. 1302. These ρ-neighborhoods, being open, necessarily belong to the domain of definition, T, of ϕ. Cf. J. Radon, *ibid.*, p. 1308. The conditions of Theorem III of the present paper, being analogous to the du Bois-Reymond condition for Riemann-integrability, may be shown to be equivalent to conditions corresponding to those of Lebesgue for the existence of the Riemann integral. We use the former set of necessary and sufficient conditions as better suited for our applications.

† Cf. J. Radon, *loc. cit.*, I, p. 1303.

It follows from the definition of saltus that

$$\omega(\xi,\eta) \leqq U_\delta(\xi,\eta) - L_\delta(\xi,\eta) < \omega(\xi,\eta) + \epsilon,$$

provided $\delta \leqq \delta' = \delta'(\epsilon;\xi,\eta)$. Consequently, by the Heine-Borel Theorem, J is covered by a finite number, N, of open squares R_i: $|\xi_i - x| < \delta_i$, $|\eta_i - y| < \delta_i$ such that the fluctuation of $f(x,y)$ in R_i is $< \omega(\xi_i,\eta_i) + \epsilon$, where (ξ_i,η_i) is a definite point interior to R_i. No restriction is imposed by requiring that all R_i be less in diameter than a fixed arbitrarily preassigned quantity ζ. The sides of the rectangles R_i, if continued to meet the sides of J, divide J into a finite number of rectangles. Let d be the length of the smallest side to be found among these resulting rectangles. Any set E_k in J whose diameter is $< d$ is interior to one of the squares R_i, so that in E_k the fluctuation of $f(x,y)$ lies between $\omega(\xi_i,\eta_i)$ and $\omega(\xi_i,\eta_i) + \epsilon$, where (ξ_i,η_i) is some point of a square R_i of diameter $< \zeta$ which contains E_k in its interior. If $\zeta < \rho$, the R_i containing any E_k'' is such that $\omega(\xi_i,\eta_i) < \epsilon$. Accordingly, the E_k'' can be taken so fine that in each $U_k - L_k < 2\epsilon$, in which case

$$0 \leqq u''(\Pi) - l''(\Pi) \leqq 2\epsilon \int\int_{E_0} d\phi(E) \leqq 2\epsilon\phi(J).$$

Again,

$$0 \leqq u'(\Pi) - l'(\Pi) \leqq 2B\phi(\Delta_{\epsilon,2\rho}) < 2B\zeta',$$

where $B = \text{fin sup}\,|f(x,y)|$ in E_0.* It follows that

$$0 \leqq u(\Pi) - l(\Pi) < \epsilon', \quad \text{where} \quad \epsilon' = 2(\epsilon\phi(J) + B\zeta'),$$

provided $D(\Pi) < d < \rho$, the order of choice of the various quantities being as follows:

Given ϵ', we take $\epsilon \leqq \epsilon'/[4\phi(J)]$. This value of ϵ determines the region G_ϵ. Then ζ' is chosen $\leqq \epsilon'/[4B]$ whereby ρ is determined, and thereupon ζ. Finally, ϵ and ζ being known, d can be determined. It has therefore been shown that as $D(\Pi) \to 0$, $\lim \sum_{k=1}^{m} U_k\phi(E_k)$ and $\lim \sum_{k=1}^{m} L_k\phi(E_k)$ exist and are equal.

This limiting value is independent of the special choice of the sequence of divisions $\Pi_1, \Pi_2, \cdots$, since if $\Pi'_1, \Pi'_2, \cdots$ is a second sequence of divisions such that $D(\Pi'_i) \to 0$, the two divisions lead respectively to partial sums of the form

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* "fin sup" is used to denote the least upper bound. Cf. E. K. Haviland, *loc. cit.*, I, note 15.

$$S_m(\Pi) = \sum_{k=1}^{m} f(x_k, y_k)\phi(E_k) \quad \text{and} \quad S_{m'}(\Pi') = \sum_{l=1}^{m'} f(x'_l, y'_l)\,\phi(E'_l).$$

But

$$|S_m(\Pi) - S_{m'}(\Pi')| = \Big|\sum_{k=1}^{m}\sum_{l=1}^{m'} [f(x_k, y_k) - f(x'_l, y'_l)]\phi(E_k \cdot E'_l)\Big|$$
$$\leqq \vartheta(D(\Pi) + D(\Pi'))\phi(J) + 2B\phi(\Delta_{\epsilon,2\rho}),$$

where $\vartheta(d) = \text{fin sup}\,|f(x,y) - f(x',y')|$ for all point pairs (x,y), (x',y') in $J - \Delta_{\epsilon,\rho}$ whose distance is $< d$. Hence $|S_m(\Pi) - S_{m'}(\Pi')| < \frac{1}{2}\epsilon_1$ if $D(\Pi) < \delta_1$, $D(\Pi') < \delta'_1$, say. But

$$|S_m(\Pi) - \int\int_{E_0} f(x,y)\,d_{xy}\phi(E)| < \tfrac{1}{2}\epsilon_1 \quad \text{if} \quad D(\Pi) < \delta_2, \text{ say.}$$

Consequently, taking $D(\Pi) < \min(\delta_1, \delta_2)$ one obtains

$$|S_{m'}(\Pi') - \int\int_{E_0} f(x,y)\,d_{xy}\phi(E)| < \epsilon_1, \quad \text{if} \quad D(\Pi') < \delta'_1.$$

Hence as $D(\Pi') \to 0$,

$$\lim \sum_{l=1}^{m'} f(x'_l, y'_l)\phi(E_l) = \int\int_{E_0} f(x,y)\,d_{xy}\phi(E), \qquad \text{q. e. d.}$$

This concludes the proof of the sufficiency of the condition.

To show the necessity of the condition, it is sufficient to prove that if the condition is not fulfilled, $\int\int_J f(x,y)\,d_{xy}\phi(E)$ fails to exist. The proof is as follows:

Let J be divided by a network Σ_n into squares

$$S_{ij}^{(n)} : (M(i-1)2^{-n} \leqq x < Mi2^{-n};\ M(j-1)2^{-n} \leqq y < Mj2^{-n});$$
$$(i, j = -2^n + 1, \cdots, 2^n - 1).$$
$$S_{ij}^{(n)} : (M(2^n - 1)2^{-n} \leqq x \leqq M;\ M(j-1)2^{-n} \leqq y < Mj2^{-n});$$
$$(i = 2^n,\ j = -2^n + 1, \cdots, 2^n - 1).$$
$$S_{ij}^{(n)} : (M(i-1)2^{-n} \leqq x < Mi2^{-n};\ M(2^n - 1)2^{-n} \leqq y \leqq M);$$
$$(j = 2^n, i = -2^n + 1, \cdots, 2^n - 1).$$
$$S_{ij}^{(n)} : (M(2^n - 1)2^{-n} \leqq x \leqq M;\ M(2^n - 1)2^{-n} \leqq y \leqq M);$$
$$(i = j = 2^n).$$

where $n = 0, 1, 2, \cdots$.

Let G_ϵ, as above, represent the set of points of J at which the saltus of $f(x,y) \geqq \epsilon$. There are at most a finite number of intersections within J of the lines of the network Σ_n at which points of G_ϵ are located. About each such intersection we draw a closed square D_p of length of side $< M2^{-n}$ and such that of the four squares of Σ_n in which D_p lies, each contains points of

G_ϵ not on the boundary of D_p or else contains no points of G_ϵ at all. Consider those regions $R_{ij}^{(n)}$ obtained by subtracting the D_p from the $S_{ij}^{(n)}$ which contain points of G_ϵ. If such an $R_{ij}^{(n)}$ contains points of G_ϵ only on its boundaries, such a point, say P_{ij}, may be made an interior point of $Q_{ij}^{(n)}$, the sum of $R_{ij}^{(n)}$ and a closed circular disc about P_{ij}, provided P_{ij} does not lie on a boundary of J. This disc may be made so small as not to deprive the adjacent $R_{ij}^{(n)}$ of points of G_ϵ. In this way, G_ϵ may be covered by a non-overlapping region $V_n = \Sigma\Sigma Q_{ij}^{(n)}$ so that

(a) The diameter of every $Q_{ij}^{(n)}$ is $\leqq 2^{-n}M(5)^{\frac{1}{2}}$;

(b) Each $Q_{ij}^{(n)}$ contains a point of G_ϵ in its interior or else on a boundary of $Q_{ij}^{(n)}$ that is also a part of the boundary of J.

Consequently $U_{ij} - L_{ij} \geqq \epsilon$ in each $Q_{ij}^{(n)}$, while by hypothesis $|\phi(V_n)| > K > 0$, K independent of n.

Therefore to every $\delta > 0$ there corresponds a subdivision Π such that $D(\Pi) < \delta$, but $u(\Pi) - l(\Pi) > \epsilon K > 0$, so $\int\int_J f(x,y)\,d_{xy}\phi(E)$ does not exist, q. e. d.

We proceed to apply Theorem III to the special case of distribution functions, for which we state

LEMMA II. *If $\phi_1(E)$ and $\phi_2(E)$ be distribution functions, then there exists an everywhere dense* sequence of rectangles R_n such that the Radon integral*†

$$\Phi(Q) = \int\int_S \phi_1(Q - P_{xy})\,d_{xy}\phi_2(E)$$

exists for $Q = R_n$, $(n = 1, 2, 3, \cdots)$.

Proof. We prove the proposition first for the case where the integration is over an arbitrarily large closed finite rectangle J by showing that Theorem III is applicable. In what follows, those lines parallel to the coördinate axes containing discontinuities of a point function $f(x,y)$ will be referred to as exceptional lines of the function f, while by the singular lines of a set function $\phi(E)$ are meant, as before, the discontinuity lines parallel to the axes of the corresponding point function‡ $F(x,y)$. In the present case, we note that

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* Cf. Definition 4.

† P_{xy} is the point with coördinates (x,y). $Q - P_{xy}$ is to be taken in the sense of vector addition and therefore represents the set of points which may be represented in the form $(\xi - x,\ \eta - y)$, where (ξ,η) is a point of Q. Cf. H. Bohr and B. Jessen, *loc. cit.*, p. 332.

‡ Cf. J. Radon, *loc. cit.*, II, p. 1093.

the exceptional lines of the point function $f(x, y) = \phi_1(Q - P_{xy})$, where Q is a fixed rectangle, are those lines $x = x_i$, $y = y_j$ such that at least one side of $Q - P_{xy}$ lies on one of the singular lines $x = \xi'_i$, $y = \eta'_j$ of $\phi_1(E)$. As Q has but four sides and as the singular lines of $\phi_1(E)$ are at most denumerable,* it follows that so are the exceptional lines of the point function $f(x, y) = \phi_1(Q - P_{xy})$. Moreover, as $\phi_1(E)$ is bounded and monotone, the points at which its saltus,† i. e., the saltus of the corresponding point function, $\geqq \epsilon$ are confined to a fixed number $n = n(\epsilon)$ of its singular lines. Accordingly, the number of lines containing at least one point at which the saltus of $\phi_1(Q - P_{xy}) \geqq \epsilon$ is finite, $= 2n(\epsilon)$, for any given rectangle Q.

If the singular lines of $\phi_2(E)$ be denoted by $x = \xi_p$, $y = \eta_q$, the number of sides of all rectangles Q for which $x_i = \xi_p$, $y_j = \eta_q$, $(i, j, p, q = 1, 2, \cdots)$ is clearly at most denumerable. For they are the lines $x = \xi_p + \xi'_i$, $(i, p = 1, 2, \cdots)$ and $y = \eta_q + \eta'_j$, $(j, q = 1, 2, \cdots)$. Consequently, there is an everywhere dense set of rectangles R_n such that the exceptional lines of $f(x, y) = \phi_1(R_n - P_{xy})$ are distinct from the singular lines of ϕ_2. When, however, the exceptional lines of a point function $f(x, y)$ are at most denumerable and are such that the points at which the saltus of the function $\geqq \epsilon$ are confined to a finite number $2n(\epsilon)$ of the lines, and when none of these exceptional lines coincide with any of the singular lines of the absolutely additive set function $\phi_2(E)$, the $2n(\epsilon)$ lines can be imbedded in strips so narrow that if W denote the sum of the portions of the strips in J, the value of $\phi_2(W)$ can be made arbitrarily small, while at all points of $J - W$ the saltus of $f(x, y)$ is less than ϵ. Thus by Theorem III

$$\Phi(Q) = \int\int_J \phi_1(Q - P_{xy})\, d_{xy}\phi_2(E) \text{ exists for } Q = R_n \quad (n = 1, 2, \cdots).$$

From the definition of distribution functions, it follows that if J_1 be an arbitrarily large rectangle and J_2 a rectangle containing J_1 in its interior, then

$$\int\int_{J_1} \phi_1(R_n - P_{xy})\, d_{xy}\phi_2(E) \leqq \int\int_{J_2} \phi_1(R_n - P_{xy})\, d_{xy}\phi_2(E) \leqq \int\int_S d_{xy}\phi_2(E) = 1.$$

Consequently,

$$\Phi(R_n) = \int\int_S \phi_1(R_n - P_{xy})\, d_{xy}\phi_2(E) \quad \text{exists,} \quad (n = 1, 2, \cdots), \quad \text{q. e. d.}$$

The R_n for which Φ is defined include rectangles having infinite sides

* Cf. J. Radon, *loc. cit.*, II, p. 1093.

† Cf. Definition 6.

by virtue of the definition of ϕ_1. In particular, they include a sequence of rectangles $R_s : (-\infty < x < x_s;\ -\infty < y < y_s)$ such that the points (x_s, y_s) are everywhere dense in the plane. Moreover, these points are everywhere dense on the parallels to the axes through any one of them and they include the vertices of a set of rectangles everywhere dense in the plane. Then to Φ, defined on the R_n, we may assign a point function F^* defined only at the points (x_s, y_s) by the relation

$$F^*(x_s, y_s) = \Phi(R_s);\ R_s : (-\infty < x < x_s;\ -\infty < y < y_s).$$

On the points (x_s, y_s), we shall prove that $F^*(x_s, y_s)$ possesses the following properties:

$$(\alpha)\quad F^*(x+h, y+k) - F^*(x+h, y) - F^*(x, y+k) + F^*(x, y) \geqq 0,$$

where $h \geqq 0$, $k \geqq 0$ and $(x+h, y+k)$, $(x+h, y)$, $(x, y+k)$, (x, y) are points of the everywhere dense sequence $\{(x_s, y_s)\}$.

$$(\beta)\qquad \lim_{h=0,\,k=0} F^*(x-h, y-k) = F^*(x, y),$$

where $h \geqq 0$, $k \geqq 0$ and (x, y), $(x-h, y-k)$ belong to the everywhere dense sequence.

$$(\gamma)\quad F^*(-\infty, y) = F^*(x, -\infty) = 0 \quad \text{and} \quad F^*(+\infty, +\infty) = 1,$$

where $F^*(-\infty, y) = \lim_{x=-\infty} F^*(x, y)$, the (x, y) being points of the everywhere dense sequence, and similar interpretations are to be placed on $F^*(x, -\infty)$ and $F^*(+\infty, +\infty)$.

Proof of (α). Let us denote $x+h$, x, $y+k$, y by x_2, x_1, y_2, y_1 respectively, and the rectangle $(-\infty < x < x_i;\ -\infty < y < y_j)$ by R_{ij}. Then

$$\begin{aligned}
&F^*(x+h, y+k) - F^*(x+h, y) - F^*(x, y+k) + F^*(x, y)\\
&\quad = F^*(x_2, y_2) - F^*(x_2, y_1) - F^*(x_1, y_2) + F^*(x_1, y_1)\\
&\quad = \Phi(R_{22}) - \Phi(R_{21}) - \Phi(R_{12}) + \Phi(R_{11})\\
&\quad = \int\int_S [\phi_1(R_{22} - P_{xy}) - \phi_1(R_{21} - P_{xy})\\
&\qquad\qquad - \phi_1(R_{12} - P_{xy}) + \phi_1(R_{11} - P_{xy})]\, d_{xy}\phi_2(E)\\
&\quad = \int\int_S \phi_1(R - P_{xy})\, d_{xy}\phi_2(E) \geqq 0,
\end{aligned}$$

in virtue of the definition of ϕ_1 and ϕ_2, and since ϕ_1 is additive and

$$R_{22} - R_{21} - R_{12} + R_{11} = R : (x_1 \leqq x < x_2;\ y_1 \leqq y < y_2), \qquad \text{q. e. d.}$$

Proof of (β). It is to be shown that if

$$R^0: (-\infty < x < x^0; -\infty < y < y^0)$$

and

$$R_s: (-\infty < x < x_s; -\infty < y < y_s),$$

then $0 \leqq \Phi(R^0) - \Phi(R_s) < \epsilon$ provided $0 \leqq x^0 - x_s < \delta$ and $0 \leqq y^0 - y_s < \delta$, $\delta = \delta(\epsilon; x^0, y^0)$, where R^0, R_s are rectangles of the everywhere dense set for which Φ is defined. We show this first for

$$\Phi_J(Q) = \int\int_J \phi_1(Q - P_{xy})\, d_{xy}\phi_2(E),$$

where J is a finite rectangle.

Let W be a closed rectangle so large that it contains the points (u, v), where $u = x^0 - x$, $v = y^0 - y$ for all (x, y) in J. With ϕ_1 there is associated a point function $F_1(u, v)$ such that

$$|F_1(u, v) - F_1(u', v')| < \tfrac{1}{2}\epsilon, \quad \text{provided} \quad \begin{aligned} |u - u'| &< \delta(\epsilon; u, v), \\ |v - v'| &< \delta(\epsilon; u, v), \end{aligned}$$

for all points (u, v) of the plane (hence *a fortiori* of W) except at most the points of a finite number $\mu = \mu(\epsilon)$ of lines $u = u_i$ and a finite number $\nu = \nu(\epsilon)$ of lines $v = v_j$. To these there correspond $\mu(\epsilon)$ lines $x = \xi_i$ and $\nu(\epsilon)$ lines $y = \eta_j$ which by the choice of R^0 are non-singular lines of $\phi_2(E)$, so that they may be imbedded in a finite number of open strips X_i, Y_j so narrow that

$$0 \leqq \sum_{i=1}^{\mu} \phi_2(X_i) + \sum_{j=1}^{\nu} \phi_2(Y_j) < \epsilon/4.$$

To these strips there correspond strips U_i containing the u_i and V_j containing the v_j. By reasoning similar to that used in Theorem III, it follows that in the closed region

$$W(S - (\sum_{i=1}^{\mu} U_i \dotplus \sum_{j=1}^{\nu} V_j)) = W_1, \qquad \delta(\epsilon; u, v) > \delta'(\epsilon) > 0,$$

where $\delta'(\epsilon)$ is independent of the point (u, v). Hence

$$|F_1(x^0 - x, y^0 - y) - F_1(x_s - x, y_s - y)| < \tfrac{1}{2}\epsilon,$$

so that $|\phi_1(R^0 - P_{xy}) - \phi_1(R_s - P_{xy})| < \tfrac{1}{2}\epsilon$ provided $|x^0 - x_s| < \delta'(\epsilon)$ and $|y^0 - y_s| < \delta'(\epsilon)$, where $\delta'(\epsilon)$ is independent of $(x^0 - x, y^0 - y)$ in W_1, and therefore of (x, y) in $J(S - (\sum_{i=1}^{\mu} X_i \dotplus \sum_{j=1}^{\nu} Y_j)) = J_1$. Consequently,

$$\left| \int\int_{J_1} [\phi_1(R^0 - P_{xy}) - \phi_1(R_s - P_{xy})]\, d_{xy}\phi_2(E) \right| \leqq \tfrac{1}{2}\epsilon \int\int_{J_1} d_{xy}\phi_2(E) \leqq \tfrac{1}{2}\epsilon.$$

Moreover,

$$\left| \int\int_{J-J_1} [\phi_1(R^0 - P_{xy}) - \phi_1(R_s - P_{xy})]\, d_{xy}\phi_2(E) \right| \leqq 2 \int\int_{J-J_1} d_{xy}\phi_2(E) \leqq 2[\sum_{i=1}^{\mu} \phi_2(X_i) + \sum_{j=1}^{\nu} \phi_2(Y_j)] < \tfrac{1}{2}\epsilon,$$

so

$$\left| \int\int_{J} [\phi_1(R^0 - P_{xy}) - \phi_1(R_s - P_{xy})]\, d_{xy}\phi_2(E) \right| < \epsilon.$$

Now suppose J so large that

$$\int\int_{S-J} d_{xy}\phi_2(E) < \tfrac{1}{2}\epsilon$$

and let R^0 : $(-\infty < x < x^0; -\infty < y < y^0)$ be a fixed rectangle for which Φ is defined. Then if R_s : $(-\infty < x < x_s;\ -\infty < y < y_s)$ be another rectangle for which Φ is defined and such that

$$|x^0 - x_s| < \delta'(\epsilon), \qquad |y^0 - y_s| < \delta'(\epsilon),$$

$$\begin{aligned} &|F^*(x^0, y^0) - F^*(x_s, y_s)| \\ &\quad \leqq \left| \int\int_{S-J} [\phi_1(R^0 - P_{xy}) - \phi_1(R_s - P_{xy})]\, d_{xy}\phi_2(E) \right| \\ &\quad + \left| \int\int_{J} [\phi_1(R^0 - P_{xy}) - \phi_1(R_s - P_{xy})]\, d_{xy}\phi_2(E) \right| < 2\epsilon. \end{aligned}$$

Hence $F^*(x_s, y_s)$ possesses property (β), q. e. d.

Proof of (γ). To prove the first part of (γ), we note that by choosing a finite rectangle J so large that $\int\int_{S-J} \phi_1(R_s - P_{xy})\, d_{xy}\phi_2(E) < \tfrac{1}{2}\epsilon$ and then choosing R_s : $(-\infty < x < x_s;\ -\infty < y < y_s)$ so distant that $\phi_1(R_s - P_{xy}) < \tfrac{1}{2}\epsilon$ for all P_{xy} in J, a result which can be brought about by taking $x_s < -M(\epsilon)$ or $y_s < -M(\epsilon)$, by virtue of the definition of ϕ_1 as a distribution function, we obtain for such a rectangle R_s

$$F^*(x_s, y_s) = \int\int_{S} \phi_1(R_s - P_{xy})\, d_{xy}\phi_2(E) < \epsilon. \qquad \text{q. e. d.}$$

The proof of the second part of (γ) is as follows: *

* Another proof of this statement is possible with the use of Fourier transforms, in virtue of Theorem V below.

$$\int\int_S \phi_1(R_s - P_{xy})\, d_{xy}\phi_2(E) \geqq \int\int_K \phi_1(R_s - P_{xy})\, d_{xy}\phi_2(E),$$

where K is a fixed finite rectangle so large that $\int\int_K d_{xy}\phi_2(E) > 1 - \epsilon$. R_s may then be taken so large, say $x_s > M(\epsilon)$, $y_s > M(\epsilon)$, that $\phi_1(R_s - P_{xy}) > 1 - \epsilon$ for all P_{xy} in K. Then

$$\int\int_S \phi_1(R_s - P_{xy})\, d_{xy}\phi_2(E) > (1-\epsilon)^2, \text{ i. e. } F^*(+\infty, +\infty) = 1, \text{ q. e. d.}$$

Now it is known † that if a function F^*, defined for a set of points everywhere dense in the plane and which are also such that they are everywhere dense on the parallels to the axes through any one of them and that they include the corners of an everywhere dense set of rectangles, possesses properties (α), (β), (γ), then there exists at least one function F defined in the whole plane and such that F fulfills conditions (α), (β), (γ) there and $F = F^*$ on all points of the everywhere dense set. To this function F there corresponds ‡ a monotone absolutely additive set function $\psi(E)$ such that $\psi(S) = 1$. Furthermore, F is determined by F^* up to the singular lines of F. We have thus proved

THEOREM IV. *Corresponding to every pair of distribution functions ϕ_1, ϕ_2 there exists a distribution function $\psi(E)$ such that* §

$$\psi(R) = \int\int_S \phi_1(R - P_{xy})\, d_{xy}\phi_2(E)$$

for an everywhere dense set of rectangles R. Furthermore, ψ is uniquely determined, up to its singular lines, by ϕ_1 and ϕ_2.

An important property of $\psi(E)$ is given by

THEOREM V.

$$\Lambda(s, t; \psi) = \Lambda(s, t; \phi_1) \cdot \Lambda(s, t; \phi_2),$$

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† Cf. J. Radon, *loc. cit.*, I, pp. 1338-1339. For our present purposes, it is necessary to make slight modifications in the reasoning because of the definition of F^* in the entire plane and to add the condition that as $h \to 0$, $k \to 0$, $\lim F^*(x - h, y - k) = F^*(x, y)$, where $(x - h, y - k)$ and (x, y) are points of the everywhere dense set for which F^* is defined.

‡ Cf. J. Radon, *loc. cit.*, I, pp. 1305-1313. Slight modifications are necessary in the case of functions defined throughout the plane.

§ This relation is often stated and used in its general form but without indication of a method of proof. Cf. e. g., R. v. Mises, *op. cit.*, p. 219.

Theorems IV and V of the present paper have been proved for the case of damped distribution functions by the author, *loc. cit.*, III.

where Λ *is the Fourier transform as introduced in* (1).

Proof. Let J be a rectangle so large that

$$0 \leqq \int\int_{S-J} d_{xy}\sigma(E) < \epsilon/4; \qquad \sigma = \psi, \phi_1, \phi_2. \tag{15}$$

As all three integrals in the statement of the theorem clearly exist, they are by their definition as improper Radon integrals limits of approximating sums taken over suitably large rectangles, in which the E_k may be chosen as rectangles R_k of the everywhere dense set for which Φ is defined. By choosing the R_k sufficiently small and $R = \sum_{k=1}^{N} R_k$ sufficiently large that it encloses J, we may obtain

$$| \Lambda(s,t;\psi) - \sum_{k=1}^{N} \exp[i(sx_k + ty_k)]\psi(R_k)| < \tfrac{1}{2}\epsilon \tag{16}$$

and if in addition R be chosen so large that $R - P_j{}^* = \sum_{k=1}^{N} (R_k - P_j)$ includes J for all points P_j in J, we have for a sufficiently fine mesh

$$| \sum_{k=1}^{N} \exp[i\{s(x_k - x_j) + t(y_k - y_j)\}]\phi_1(R_k - P_j) - \int\int_{R-P_j} \exp[i(sx+ty)]d_{xy}\phi_1(E)| < 3\epsilon/4.$$

Also

$$| \Lambda(s,t;\phi_1) - \int\int_{R-P_j} \exp[i(sx+ty)]d_{xy}\phi_1(E)| < \epsilon/4,$$

so that

$$| \Lambda(s,t;\phi_1) - \sum_{k=1}^{N} \exp[i\{s(x_k - x_j) + t(y_k - y_j)\}]\phi_1(R_k - P_j)| < \epsilon.$$

In other words,

$$\sum_{k=1}^{N} \exp[i\{s(x_k - x_j) + t(y_k - y_j)\}]\phi_1(R_k - P_j) = \Lambda(s,t;\phi_1) + \zeta_{1j}, \tag{17}$$

where $|\zeta_{1j}| < \epsilon$, provided R is sufficiently large and its subdivision sufficiently fine. It is to be noted that the fineness of the R_k is independent of P_j, as is also ϵ, so that $N = N(\epsilon)$ is fixed. The meshes may then be further subdivided, if necessary, so that

$$| \sum_{k=1}^{N} \{\psi(R_k) - \sum_{j=1}^{m} \phi_1(R_k - P_j)\phi_2(E_j)\} | < \epsilon/2,$$

* This represents vector addition as described in the second footnote on p. 646, R corresponding to Q and P_j to P_{xy}.

where $m = m(\epsilon)$ and $\sum_{j=1}^{m} E_j$ need not extend beyond J. On multiplying the expression within the absolute value signs by $\exp[i(sx_k + ty_k)]$ and interchanging the order of the double summation, we have with the help of (16):

$$(18)\quad |\Lambda(s,t;\psi) - \sum_{j=1}^{m}\sum_{k=1}^{N} \exp[i\{s(x_k - x_j) + t(y_k - y_j)\}]\phi_1(R_k - P_j) \cdot \exp[i(sx_j + ty_j)]\phi_2(E_j)| < \epsilon.$$

Let the E_j be so fine (if this is not already the case) and $K = \sum_{j=1}^{m} E_j$ so large that

$$(19)\quad \sum_{j=1}^{m} \exp[i(sx_j + ty_j)]\phi_2(E_j) = \Lambda(s,t;\phi_2) + \zeta_2, \quad \text{where} \quad |\zeta_2| < \epsilon.$$

Substituting (17) and (19) in (18), we obtain

$$|\Lambda(s,t;\psi) - \Lambda(s,t;\phi_1)\cdot\Lambda(s,t;\phi_2) - \zeta_2\Lambda(s,t;\phi_1) - \sum_{j=1}^{m}\zeta_{1j}\exp[i(sx_j + ty_j)]\phi_2(E_j)| < \epsilon.$$

But by virtue of the appraisals for ζ_2 and ζ_{1j} and the fact that the total variation of ϕ_1 and of ϕ_2 is 1, we obtain

$$|\zeta_2\Lambda(s,t;\phi_1)| < \epsilon$$

and

$$|\sum_{j=1}^{m}\zeta_{1j}\exp[i(sx_j + ty_j)]\phi_2(E_j)| \leqq \sum_{j=1}^{m}|\zeta_{1j}|\,\phi_2(E_j) < \epsilon.$$

Consequently,

$$|\Lambda(s,t;\psi) - \Lambda(s,t;\phi_1)\cdot\Lambda(s,t;\phi_2)| < 3\epsilon,$$

i. e., $\Lambda(s,t;\psi) = \Lambda(s,t;\phi_1)\cdot\Lambda(s,t;\phi_2)$, q. e. d.

Since distribution functions are determined up to their singular lines by their Fourier transforms, it follows from the preceding theorem that $\psi(E)$ as above defined remains unchanged by interchanging ϕ_1 and ϕ_2 in the definition of Φ. A direct proof of this proposition is also possible.

3. *The spectra of distribution functions.* With a view to investigating further the addition of distributions as expressed by the symbolical product, we now lay down the following definitions:

Definition 7. The spectrum of a distribution function $\phi(E)$ is the set of all points P such that $\phi(R) > 0$ for every open rectangle R containing P.

Definition 8. The point spectrum of the distribution function $\phi(E)$ is the set of points P such that $\phi(R) > \alpha > 0$ for every open rectangle R containing P, where $\alpha = \alpha_P$ is a positive lower bound independent of R.

We are now in a position to investigate the relations between the spectra of two distribution functions and the spectrum of their symbolical product,* making use of the agreement that if, in the vector addition of two sets of numbers, one of the sets be empty, then their sum is to be regarded as empty. We shall prove

THEOREM VI. *If $\psi(E)$ be the distribution function defined by*

$$\psi(R) = \int\int_S \phi_1(R - P_{xy})\, d_{xy}\phi_2(E)$$

for all rectangles R for which the integral exists, and if $S(\phi)$ denotes the spectrum of ϕ and $P(\phi)$ the point spectrum of ϕ, then the following relations hold:

$$\text{(a)} \quad S(\psi) = S(\phi_1) + S(\phi_2).$$
$$\text{(b)} \quad P(\psi) = P(\phi_1) + P(\phi_2).$$

Proof. As the proof of (a) follows the one-dimensional proof given by Wintner step by step, it is unnecessary to repeat it here, for the translation of the proof to two dimensions offers no difficulties. It is to be noted, however, that in the two dimensional case one cannot obtain a necessary and sufficient condition for the existence of the symbolical product in terms of the point spectra of ϕ_1 and of ϕ_2. For the purposes of the proof, though, it is sufficient that the integral definition of the symbolical product be valid for a set of rectangles everywhere dense.

The first step in the proof of (b) is to show that if a point (ξ, η) cannot be written in the form $(x_1, y_1) + (x_2, y_2)$, where (x_1, y_1) belongs to the point spectrum of ϕ_1 and (x_2, y_2) to the point spectrum of ϕ_2, then (ξ, η) cannot belong to the point spectrum of ψ; i. e. to any preassigned $\epsilon > 0$ there corresponds a rectangle R_ϵ containing (ξ, η) and such that $\psi(R_\epsilon) < \epsilon$.

We first consider the point spectrum of ϕ_1. Since ϕ_1 is monotone and $\phi_1(S) = 1$, it follows that the number of points $(\bar{x}_1, \bar{y}_1)$ in the plane such that $\phi_1(R) > \epsilon/2$ for every rectangle R about $(\bar{x}_1, \bar{y}_1)$ is finite, say $n = n(\epsilon/2)$. Consequently, for a fixed (ξ, η) there are only $n(\epsilon/2)$ points (x', y') such that $\phi_1(R - P_{x'y'}) > \epsilon/2$ for all R containing (ξ, η) and these are the points for which $(\xi, \eta) - (x', y') = (\bar{x}_1, \bar{y}_1)$ or $(\xi, \eta) = (\bar{x}_1, \bar{y}_1) + (x', y')$. It follows

* In the one dimensional case, these relations have been derived by A. Wintner, *loc. cit.*, V.

by hypothesis that (x', y') is not a point (x_2, y_2) of the point spectrum of ϕ_2. Consequently, about each of the $n(\epsilon/2)$ points (x'_i, y'_i), $[i = 1, 2, \cdots, n = n(\epsilon/2)]$, it is possible to take a rectangle R_i such that $\sum_{i=1}^{n} \phi_2(R_i) < \epsilon/4$. Moreover, let L be a rectangle so large that $\int\int_{S-L} d_{xy}\phi_2(E) < \epsilon/4$ and let $L - \sum_{i=1}^{n} R_i = L - U = L_1$. Then for all points (x, y) in L_1, we have $\phi_1(R - P_{xy}) < \epsilon/2$ provided R be sufficiently small. As (x, y) moves about L_1 (which may be taken as closed by supposing L closed, the R_i open), $(\xi, \eta) - (x, y) = (u, v)$ moves about a closed region L_2 which does not contain any points $(\bar{x}_1, \bar{y}_1)$ for which $\phi_1(R) > \epsilon/2$ for every R about $(\bar{x}_1, \bar{y}_1)$. Hence about every point (u, v) of L_2 it is possible to take a rectangle R_{uv} such that $\phi_1(R_{uv}) < \epsilon/2$. As L_2 is closed, it follows that there is a fixed rectangle $M(\epsilon/2)$ such that $\phi_1(R_{uv}) < \epsilon/2$ for all R_{uv} containing (u, v) and contained in a rectangle the size of $M(\epsilon/2)$. Hence if R_ϵ be a rectangle of the size of $M(\epsilon/2)$ with any (x, y) of L_1 as its centre, $\phi_1(R_\epsilon - P_{xy}) < \epsilon/2$. Now

$$\begin{aligned}\int\int_S \phi_1(R_\epsilon - P_{xy})\, d_{xy}\phi_2(E) &= \int\int_{S-L} + \int\int_{L-U} + \int\int_U \\ &\leqq \int\int_{S-L} d_{xy}\phi_2(E) + \int\int_{L_1} \phi_1(R_\epsilon - P_{xy})\, d_{xy}\phi_2(E) + \int\int_U d_{xy}\phi_2(E) \\ &< \epsilon/4 + \tfrac{1}{2}\epsilon \int\int_{L_1} d_{xy}\phi_2(E) + \epsilon/4 < \epsilon.\end{aligned}$$

Therefore, there exists a rectangle R_ϵ about (ξ, η) such that $\psi(R_\epsilon) < \epsilon$, wherefore (ξ, η) does not belong to the point spectrum of ψ. On the other hand, if $(\xi, \eta) = (x_1, y_1) + (x_2, y_2)$, the notation being as above, then (ξ, η) belongs to the point spectrum of ψ. Here, again, the proof is essentially the same as that in the one-dimensional case. This completes the proof of Theorem VI.

As has been pointed out above, the rôle played by the point spectrum in the existence of the symbolical product for the one-dimensional case is, in a sense, accidental and arises from the fact that the singular lines of the distribution functions, which in the two-dimensional case determine the existence of the symbolical product, coincide, in one dimension, with the point spectrum. We may, consequently, conclude with a few remarks on the singular lines.

In the first place, *all points of the point spectrum are necessarily situated at the intersection of two singular lines.* For it follows immediately from the definition of the point spectrum and that of the singular lines that the parallels

to the coördinate axes through a point of the point spectrum are both singular lines.

On the other hand, *not every intersection of two singular lines is a point of the point spectrum, nor even of the spectrum.* For consider the distribution function $\phi(E)$ defined as follows:

$$\phi(R) = \tfrac{1}{2} \text{ for any rectangle } R: (x_1 \leqq x < x_2;\ y_1 \leqq y < y_2)$$

containing the point $(0, 0)$ and not the point $(1, 1)$ or else containing the point $(1, 1)$ and not the point $(0, 0)$, whereas $\phi(R) = 0$ for all rectangles R not containing $(1, 1)$ or $(0, 0)$. Thus $\phi(E)$ is a distribution function. To it corresponds the point function $F(x, y)$ where $F(x, y) = 0$ if $-\infty < x \leqq 0$ or $-\infty < y \leqq 0$; $F(x, y) = \tfrac{1}{2}$ if $0 < x \leqq 1$, $0 < y < +\infty$ or $0 < x < +\infty$, $0 < y \leqq 1$; $F(x, y) = 1$ in $(1 < x < +\infty;\ 1 < y < +\infty)$. Thus $x = 0$ and $y = 1$ are singular lines, but it may easily be computed * that $\phi(R) = 0$ for all rectangles sufficiently small about $(0, 1)$, so that $(0, 1)$ does not belong to the point spectrum nor even to the spectrum.

Moreover, *singular lines may exist even if the point spectrum is vacuous.* For consider the point function $F_1(x, y)$ defined as follows:

$F_1(x, y) = 0$ if $-\infty < x \leqq 0$ or $-\infty < y \leqq 0$.
$F_1(x, y) = y$ in $(0 < x < +\infty;\ 0 < y \leqq \tfrac{1}{2})$
$F_1(x, y) = \tfrac{1}{2}$ in $(0 < x < +\infty;\ \tfrac{1}{2} < y \leqq 1)$ and in $(0 < x \leqq \tfrac{1}{2};\ 1 < y < +\infty)$
$F_1(x, y) = x$ in $(\tfrac{1}{2} < x \leqq 1;\ 1 < y < +\infty)$
$F_1(x, y) = 1$ in $(1 < x < +\infty;\ 1 < y < +\infty)$.

As $F_1(x, y)$ possesses the properties (α), (β), (γ) defined above, it determines a distribution function $\phi_1(E)$. Both $x = 0$, and $y = 1$ are seen to be singular lines of ϕ_1, but an investigation of the point spectrum shows that it is vacuous.

Finally, the proof that $F^*(x_s, y_s)$, as defined above, possesses the property (β) contains effectively the more general result that $F(x, y)\,(= \psi(E))$ is continuous at all points (x, y) where x is not of the form $\bar{x}_n = \xi_i' + \xi_j''$ and y is not of the form $\bar{y}_n = \eta_k' + \eta_l''$; the lines $x = \xi_i'$, $y = \eta_k'$ being the singular lines of ϕ_1, and $x = \xi_j''$, $y = \eta_l''$ those of ϕ_2. All points of a line $x = \bar{\bar{x}}_n \neq \bar{x}_n$ must accordingly be continuity points of F if their ordinates are not of the form $y = \bar{y}_n$. As these continuity points are thus everywhere dense on the fixed line $x = \bar{\bar{x}}_n$, every finite length of $x = \bar{\bar{x}}_n$ can be imbedded in a rectangle R such that $\psi(R) < \epsilon$. Hence $x = \bar{\bar{x}}_n$ is not a singular line of ψ and the same is true of $y = \bar{\bar{y}}_n \neq \bar{y}_n$. It follows that at most the lines

* Cf. J. Radon, *loc. cit.*, I, p. 1304.

$x = \bar{x}_n$, $y = \bar{y}_n$ are singular lines of $\psi(E)$. Again, if R be an arbitrarily narrow strip about $x = \bar{x}_n$, $R - \xi_j''$ is an arbitrarily narrow strip about $x = \xi_i'$. By hypothesis, $\phi_1(R - (\xi_j'', y))$ * $> K_1$ for all y and any R. Furthermore, $\phi_1(R - P_{xy}) > K_1 > 0$ for all x in a strip $T: \xi_j'' - \delta < x < \xi_j'' + \delta$ whose width is less than that of R. But however narrow T, $\phi_2(T) > K_2 > 0$, so $\Phi(R) > K_1 K_2 > 0$ for arbitrarily narrow R and $\bar{x}_n$ is consequently a singular line of ψ. Similarly, $\bar{y}_n$ may be shown a singular line of ψ. *Consequently, the singular lines of ψ are precisely the lines $x = \bar{x}_n$, $y = \bar{y}_n$.*

As an application of the rules obtained for the addition of spectra, we may consider the addition of convex curves as treated by H. Bohr and B. Jessen.† Then the spectrum of the distribution function associated with a given curve is precisely the curve itself.‡ By our rule, the spectrum of the distribution function belonging to the symbolical product of the distribution functions of two curves is the vector sum of the spectra of the individual curves, with the agreement that if one of the addends is the zero set, so also is the sum, a result in accordance with the definition of the distribution functon associated with the sum of the curves.§

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* This represents vector addition.

† Cf. H. Bohr and B. Jessen, *loc. cit.* An analytic treatment of the outer curve bounding the region formed in the addition of convex curves has been given by the author, *loc. cit.*, IV.

‡ Cf. H. Bohr and B. Jessen, *loc. cit.*, pp. 338-339.

§ Cf. H. Bohr and B. Jessen, *ibid.*, p. 340.

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